Introduction

For ν-sets with faces and degeneracies, the familiar coherence laws say that faces commute with faces, faces commute with degeneracies, and degeneracies commute with degeneracies. These laws can themselves be composed in different orders, producing parallel proofs of the same equality. This is a well-known higher-coherence phenomenon of weak higher-dimensional structures: once structural laws are represented by paths rather than strict equations, different composites of those laws must themselves be related by higher paths.

Bonak works in `HSet`, so these parallel proofs are identified by `UIP`; in `HGpd` one would need to prove one more level explicitly, and in unrestricted `Type` this pattern continues as an infinite tower of higher coherences. This note records the next level of these coherence equations. It is meant as a guide to the mathematical content hidden behind the `UIP` calls in Bonak, and to what would have to be made explicit in a future version of the construction beyond `HSet`.

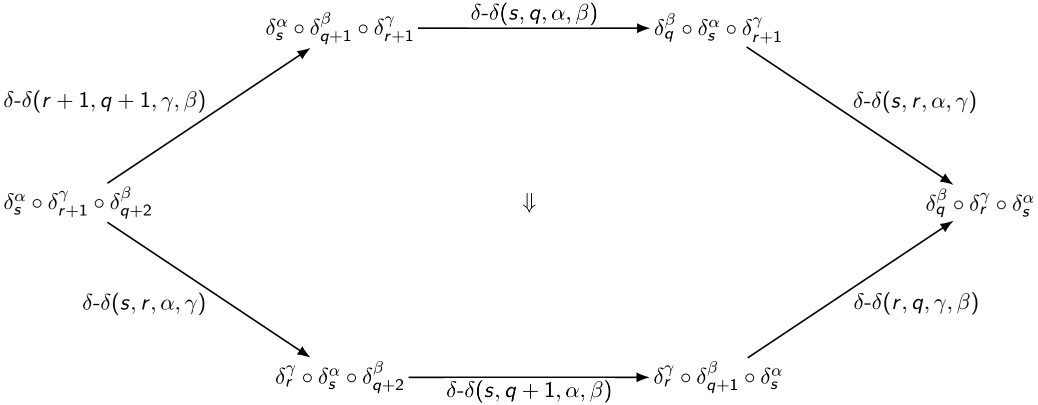
Notation

- Restrictions, i.e. faces, are denoted by δ.
- Reflexivities, i.e. degeneracies, are denoted by ε.
- Reflexivities "above" and "below" are distinguished by the superscripts $\uparrow$ and $\downarrow$.
- The superscript α, β, γ denotes the arity.
- The subscript q, r, s denotes the direction.

Coherences

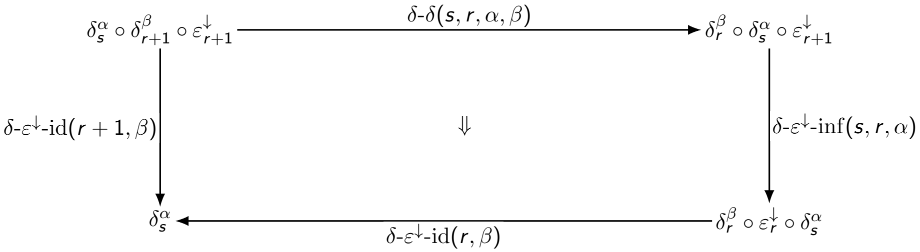
$$
\begin{aligned}
&\delta\text{-}\delta\ r\ q\ \gamma\ \beta\ (r \leq q \leq k): && \delta_r^\gamma \circ \delta_{q+1}^\beta = \delta_q^\beta \circ \delta_r^\gamma \\
&\delta\text{-}\varepsilon^\downarrow\text{-id}\ r\ \beta\ (r \leq k): && \delta_r^\beta \circ \varepsilon_r^\downarrow = \mathrm{id} \\
&\delta\text{-}\varepsilon^\downarrow\text{-inf}\ r\ q\ \beta\ (r \leq q \leq k): && \delta_r^\beta \circ \varepsilon_{q+1}^\downarrow = \varepsilon_q^\downarrow \circ \delta_r^\beta \\
&\delta\text{-}\varepsilon^\downarrow\text{-sup}\ r\ q\ \beta\ (q \leq r \leq k): && \delta_{r+1}^\beta \circ \varepsilon_q^\downarrow = \varepsilon_q^\downarrow \circ \delta_r^\beta \\
&\delta\text{-}\varepsilon^\uparrow\text{-sup}\ r\ q\ \beta\ (q \leq p)\ (r \leq k): && \delta_r^\beta \circ \varepsilon_q^\uparrow = \varepsilon_q^\uparrow \circ \delta_r^\beta \\
&\varepsilon^\downarrow\text{-}\varepsilon^\downarrow\ r\ q\ (q \leq r \leq k): && \varepsilon_{r+1}^\downarrow \circ \varepsilon_q^\downarrow = \varepsilon_q^\downarrow \circ \varepsilon_r^\downarrow \\
&\varepsilon^\downarrow\text{-}\varepsilon^\uparrow\ r\ q\ (q \leq p)\ (r \leq k): && \varepsilon_r^\downarrow \circ \varepsilon_q^\uparrow = \varepsilon_q^\uparrow \circ \varepsilon_r^\downarrow \\
&\varepsilon^\uparrow\text{-}\varepsilon^\uparrow\ r\ q\ (q \leq r \leq p): && \varepsilon_{r+1}^\uparrow \circ \varepsilon_q^\uparrow = \varepsilon_q^\uparrow \circ \varepsilon_r^\uparrow
\end{aligned}
$$

δ-δ

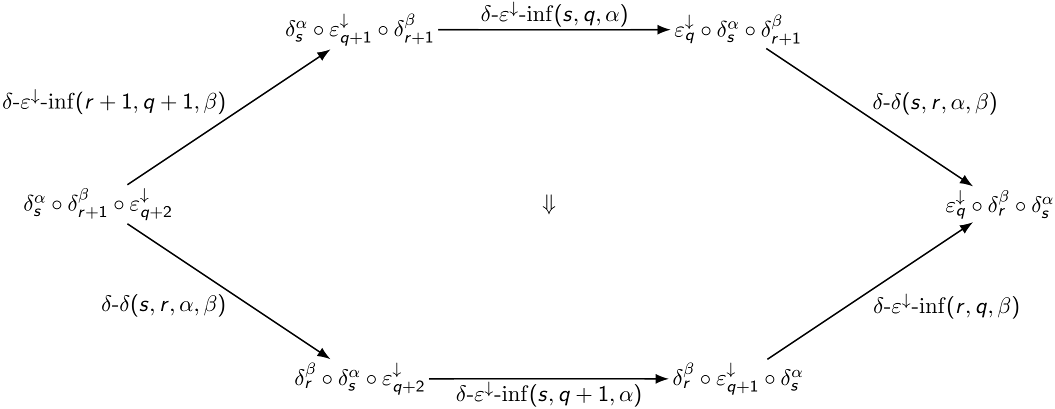


$\delta^\alpha_s \circ \delta^\beta_{q+1} \circ \delta^\gamma_{r+1}$
δ-$\delta(s, q, \alpha, \beta)$
$\delta^\beta_q \circ \delta^\alpha_s \circ \delta^\gamma_{r+1}$
δ-$\delta(r+1, q+1, \gamma, \beta)$
δ-$\delta(s, r, \alpha, \gamma)$
$\delta^\alpha_s \circ \delta^\gamma_{r+1} \circ \delta^\beta_{q+2}$
$\Downarrow$
$\delta^\beta_q \circ \delta^\gamma_r \circ \delta^\alpha_s$
δ-$\delta(s, r, \alpha, \gamma)$
δ-$\delta(r, q, \gamma, \beta)$
$\delta^\gamma_r \circ \delta^\alpha_s \circ \delta^\beta_{q+2}$
$\delta^\gamma_r \circ \delta^\beta_{q+1} \circ \delta^\alpha_s$
δ-$\delta(s, q+1, \alpha, \beta)$

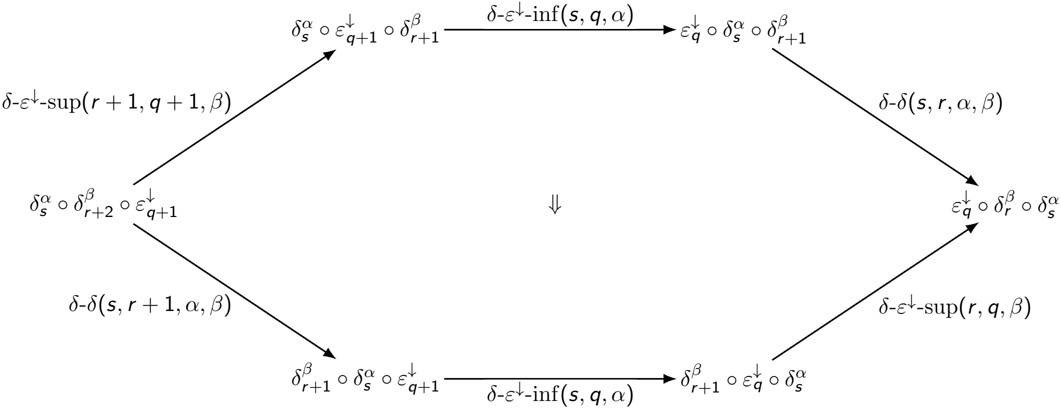
δ-$\varepsilon^{\downarrow}$-id



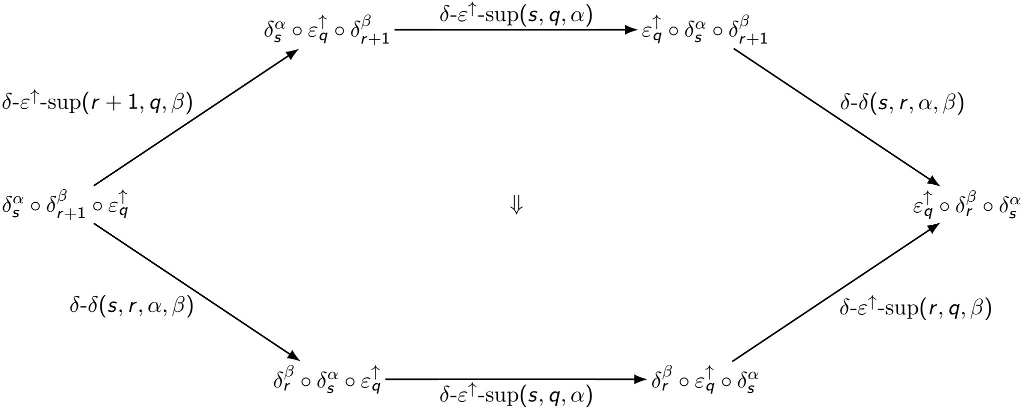
δ-$\varepsilon^{\downarrow}$-inf



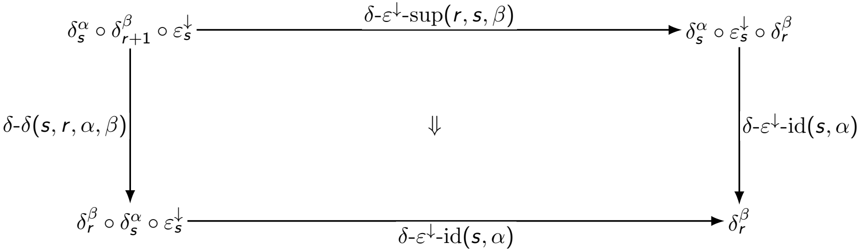
δ-$\varepsilon^{\downarrow}$-sup



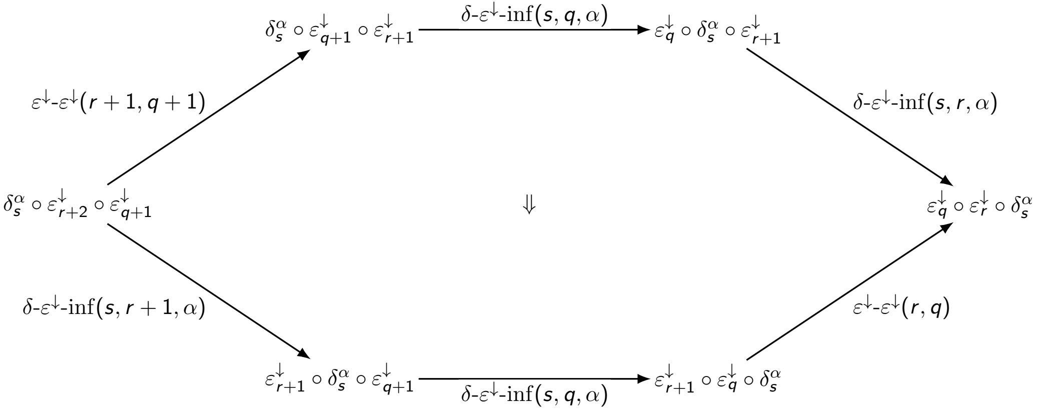
δ-$\varepsilon^{\uparrow}$-sup (S)



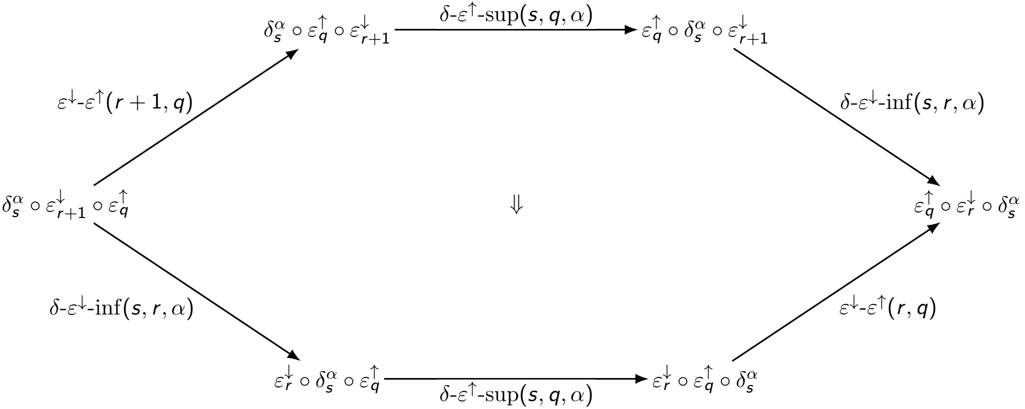
δ-$\varepsilon^{\uparrow}$-sup (0)



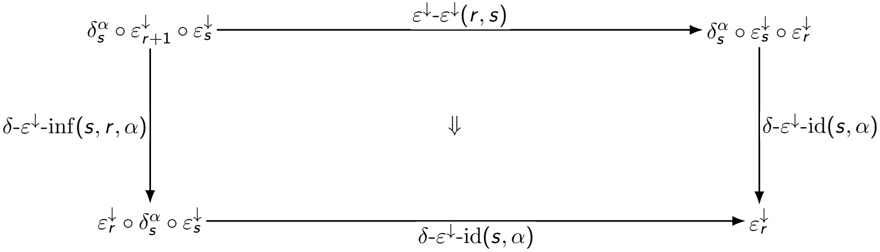
$\varepsilon^{\downarrow}$-$\varepsilon^{\downarrow}$



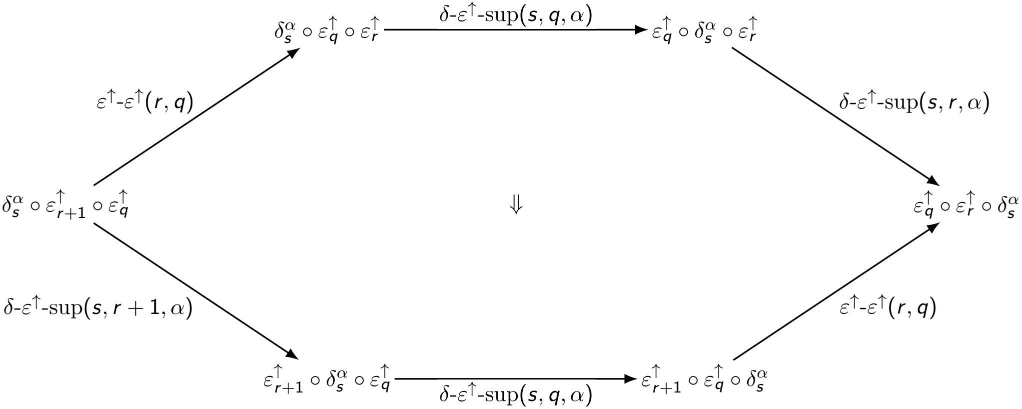
$\varepsilon^{\downarrow}$-$\varepsilon^{\uparrow}$ (S)



$\varepsilon^{\downarrow}$-$\varepsilon^{\uparrow}$ (0)



$\varepsilon^{\uparrow}$-$\varepsilon^{\uparrow}$ (S)



$\varepsilon^{\uparrow}$-$\varepsilon^{\uparrow}$ (0)